Overview

This is a quick overview of everything in Stage II
for those of you who are in a hurry and have the
mathematical background to take it all in.
Don't forget if you are a teacher there are free
teaching games, notes and exercises
which might be useful for you
on the Stage II page.

Order of Subtracting

Traditionally we subtract back to front, the opposite way we read.
There is no need for this.
It is possible to subtract in any order, and going
forwards is the obvious best choice.

The "Complement of the Difference Strategy"

We will look at a second circular strategy for getting answers when
the subtraction "won't go". From Stage I you will have met
the "Add a Complement" strategy, where for example:

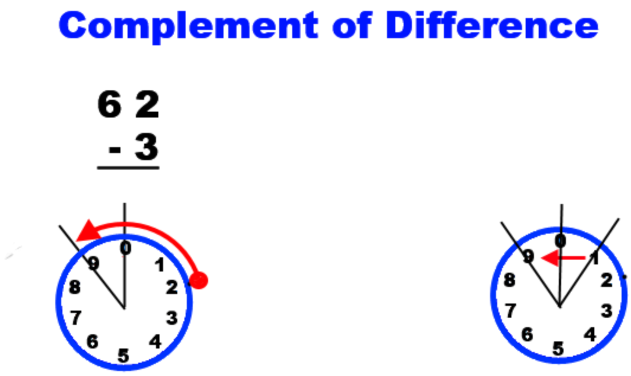
$$\begin{array}{r} 6\ 2 \\ \underline{-\ \ 3} \\ \underline{5\ 9} \end{array}$$

...by the "Add a Complement" strategy (previously covered):

2 - 3 = 2 + 7 = 9 (We add the complement of 3 which is 7 in 10-circle)

...by the "Complement of the Difference" strategy (new):

2 and 3 are 1 apart
The complement of 1 is 9 in 10-circle.

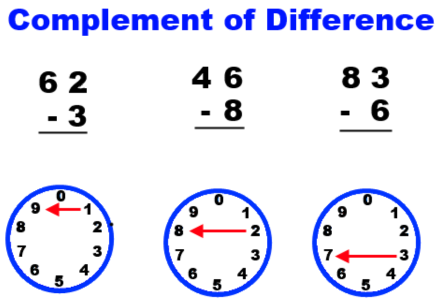


The 2-3 comes to 9 in the 10-circle as shown on the left.

Another path is to note 2 - 3 comes to -1
And taking the complement of 1 gets us to 9 which is -1

The "Complement of the Difference" strategy
is literally that, to take the complement of the difference:

Here are some Examples:



2 - 3 : Difference = 1: Complement of 1 = 9
6 - 8: Difference = 2: Complement of 2 = 8
3 - 6: Difference = 3: Complement of 3 = 7

When to Use

When should you use which strategy?

In general, use the "Complement of the Difference" strategy
when the digits subtracted are close together,
for example in 72 - 4 = 68 (2 and 4 are only 2 apart, complement 8)

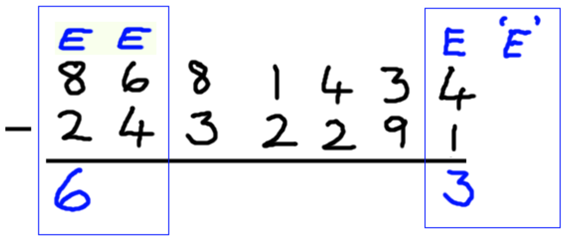
...and use the "Add a Complement" strategy
when a large number (like 7,8 or 9) is being subtracted.
for example in 73 - 8 = 65 (8 is large, it's complement is 2, add it to the 3)

Long Subtraction

There are 4 basic types of subtractions:
We group subtractions in subtraction PAIRS

The four types are:
"EE", "EH", "HE" and "HH"
where
E means "easy" meaning "subtracts to a result above zero".
H means "hard" meaning "subtracts to a result below zero".

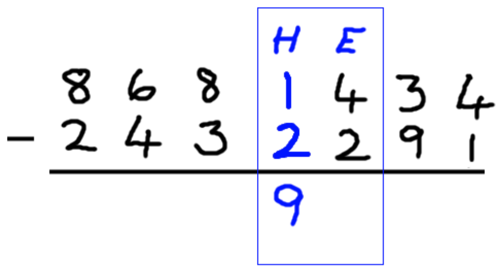
Type 1: EE ("Easy Easy")



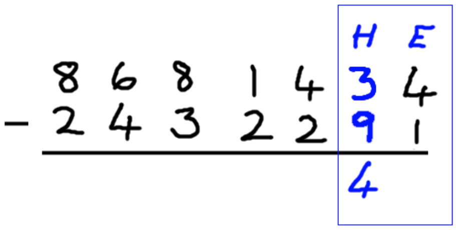
In this example 8-2 is "easy". It's 6.
Its neighbour (6-4) is also "easy". We can just put down the 6.
That's the "Easy-Easy" type.

The 4 - 1 at the end is also "easy". It has no neighbour to effect it.
It's also an "Easy-Easy" type. So put it down exactly as it is. 4 - 1 is 3.

Type 2: HE ("Hard-Easy")

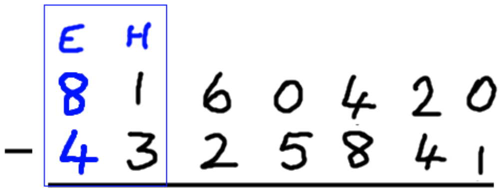


The "1 - 2" goes below zero.
We don't know what it comes to. It's hard.
Fortunately its neighbour, the "4-2", is "easy" and will have no effect on it.
We know what to do when the subtraction "won't go". Just use one of our two complement strategies.
The difference between 1 and 2 is 1, so take the complement of that and put it down as 9.
That is the "Hard-Easy" type. Just use one of the strategies.

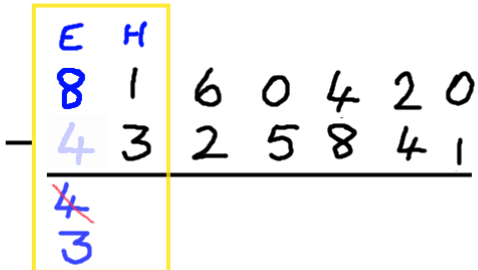


There's another example of the "Hard-Easy" type: "3 - 9"
It's hard because it goes below zero.
It's neighbour (the "4 - 1") is easy so has no effect on it.
This time I'll add the complement of 9 (which is 1) to the 3 to get the answer of 4 which I'll just put down.
Once again that's an "Hard-Easy" type. It doesn't matter which of the two strategies you use.

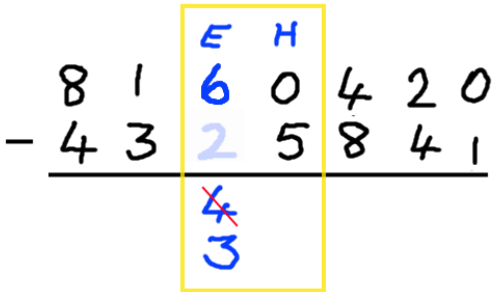
Type 3: EH ("Easy-Hard")



In this example "8 - 4" is easy but its neighbour "1-3"
goes below zero which is "hard".
This is an "Easy-Hard" type.
If it were "Easy-Easy" we would just say the answer is 4
and that would be that. But it's not.



You can see (if we mask out the "forty" part of "43")
that "81 - 3" is going to be forced into the seventies.
The "8-4" will drop down 1 from 4 to become a 3.
We don't need to know what the "1-3" comes to.
All we need to know is that it is Hard.
So reduce its neighbour by 1.



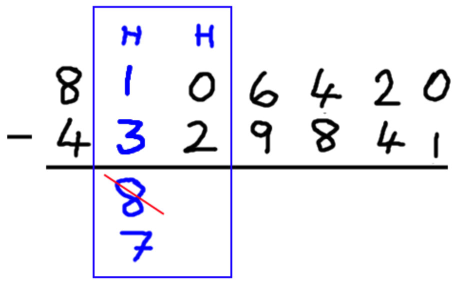
Here's another example of "Easy-Hard":
"6-2" is "easy".   It's 4.
But its neighbour "0-2" goes below zero which makes it "hard".
Again if we mask the "twenty" part out we can see that 60 - 5 will drop into the fifties.
We will then be dealing with "50 something" less "twenty something" which comes to "30 something".
We will get a 3 there instead of a 4.
Again the answer drops down
by exactly 1 when the neighbour is hard.

And that is the general rule,
which is to ***drop down by 1 if the subtraction pair on the right is "Hard"***.

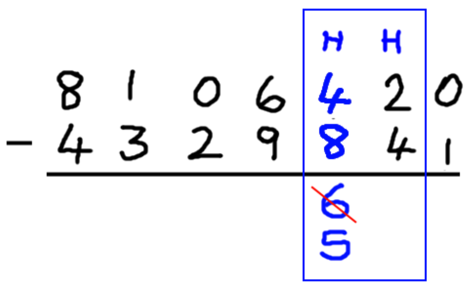
That's how we handle the "Easy-Hard" type
and it's also pretty much how we handle the "Hard-Hard" type as well,
as we will now show you:

Type 4: HH ("Hard-Hard")

Our fourth and final type is the "Hard-Hard" type.



In this example "1-3" goes below zero so it is "hard" and so is its neighbour "0 - 2".
Because it is hard we will have to handle the "1-3" using one of our strategies.
In this case the difference between 1 and 3 is 2 and it's complement is 8.
But the neighbour is hard also.
So just as before in the previous cases
we need to drop the answer down by 1.
It goes from 8 down to 7.



Another example is the "4-8" which is "hard" and its neighbour "2 - 4" which is also "hard".
We need to handle the "4 - 8" using one of our strategies,

and in this case I'll take the complement of 8 which is 2 and add it to the 4 to get 6.
But because the neighbour is hard we will reduce that to a 5.

And that's how we handle the "Hard-Hard" type.

Neither Above nor Below Zero:
Zero Exactly

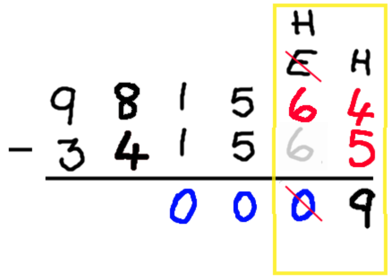
There is a fifth type.
Subtractions which are neither above nor below zero.
Subtractions which come to zero exactly.

	9	8	1	5	6	5
−	3	4	1	5	6	4
	6	4	0	0	0	1

Consider the above sum.
The middle results all subtract to zeros.
Think of it like a sandwich, with the "8-4" and the "5-4" being the bread,
with the zeros being the filling in-between.
If, as in this case,
the far right result comes out "above zero" and is "easy"
then the answer is just what you would expect.
It's 640,001.

	9	8	1	5	6	4
−	3	4	1	5	6	5
			0	0	0	9

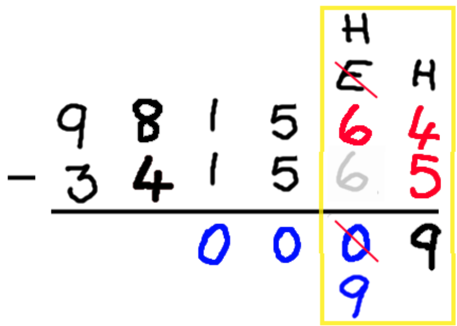
If however, we change the last digits around so that they go below zero
and form a "hard" pair (as above), then the picture is quite different.
For a start we will need to calculate the "4-5" in 10-circle
Difference = 1, complement of that = 9.



If we mask out the "sixty" of "65" we see "64 - 5"
which will drop into the fifties straight away.
This means we are then dealing with "fifty" something
less "sixty" something
which goes below zero.

Instead of "6-6" we are dealing with "5 - 6".
A hard pair.
Because its neighbour (the "4-5" far right) is hard
we need to drop its result down by 1.

What is one below zero?
Looking at the 10-circle above we see, paradoxically, it is 9.
Put another way using the "Complement of the Difference" strategy
we can see that "5 - 6" has a difference of 1, whose complement is 9.
Either way we get the blue 9 shown below:



Because the "6-6" step ended up becoming a "5-6" step
and going "below zero"
it needs to borrow from its neighbour (the "5-5")
which in turn sends IT below zero and makes it hard,
which in turn needs to borrow from its neighbour and so on...

$$
\begin{array}{rcccccc}
 & 9 & 8 & 1 & 5 & 6 & 4 \\
- & 3 & 4 & 1 & 5 & 6 & 5 \\
\hline
 & 6 & 3 & \cancel{0} & \cancel{0} & \cancel{0} & 9 \\
 & & & 9 & 9 & 9 &
\end{array}
$$

The effect propagates like a wave
and only stops when it hits the other "side of the sandwich", the "8-4".
The neighbour of "8-4" (I mean the "1-1")
became hard so the "8-4" drops by 1, from 4 down to 3.
But it does not itself go below zero.
The effect stops: "9-3" is just 6.

To summarize:
If there is a sequence of zero subtractions then look to the far right of them.
If the subtraction on the far right "piece of bread" is easy, the whole calculation is easy.

But if it isn't, work out the end digit in the usual way
(use "Add a Complement" or "Complement of Difference" methods),
turn all the 0's into 9's and
reduce the other end digit by 1.
The effect stops there.

And that concludes how to deal with this rather unusual anomaly.

You now have enough information to be able to subtract any numbers in any order
back to front, front to back,
easily and speedily (with a little practice).

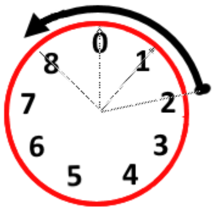
Special Refinement

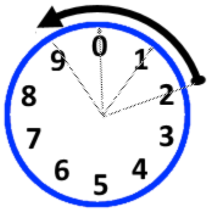
For really extra-smooth subtraction there is one further refinement you can make.
When we handle a "hard-hard" subtraction (like the "2-3" below) by:

$$
\begin{array}{rccc}
 & & H & H \\
 & 8 & 2 & 6 \\
- & & 3 & 9 \\
\hline
 & 7 & \cancel{9} & \\
 & & 8 &
\end{array}
$$

1. working it out in 10-circle then
2. reducing the result by 1

we are in fact *double handling* it.

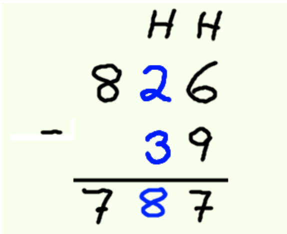




If you compare that subtraction in 10-circle ("2-3 = 9")
to the same subtraction in a 9-circle ("2-3 = 8")
you can see that the 9-circle answer is "one less" automatically.
The reduction step is done for you.

So the refinement is:

If a subtraction is "hard hard"
work it out in 9-circle
and forget about
reducing it by 1!



In our example noting the "2-3" is "Hard Hard"
we take the difference between 2 and 3 which is 1,
and then take its complement in **9**-circle.
Thats 8...

The final answer is 787.
That's all we have to do.

This might seem like a lot of work to get just a little bit of speed
but it's not as hard as you think.
Take a look at the final video/webpage in Stage II
which shows how little there is to learn to be able to do this.

Once again - if you found this difficult to follow,
take a look at the rest of the videos
which go into things in more detail and at a more comfortable pace.